PERGAMON

International Journal of Solids and Structures 38 (2001) 8563–8583

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Nonlinear vibration of an initially stressed plate based on a modified plate theory

Chun-Sheng Chen [a,*], Jiun-Ren Hwang [b], Ji-Liang Doong [b]

[a] Department of Mechanical Engineering, Nan Ya Junior College, Chung-Li 32034, Taiwan, ROC
[b] Department of Mechanical Engineering, National Central University, Chung-Li 32054, Taiwan, ROC

Received 24 August 1994; accepted 9 March 1995

Abstract

Nonlinear partial differential equations for the vibrating motion of a plate based on a modified higher order plate theory with seven kinematic variables are derived. The present seven-variable modified higher order plate theory satisfies the stress-free boundary conditions. Using these derived governing equations, the large amplitude vibrations of a simply supported thick plate subjected to initial stresses are studied. The Galerkin method is used to transform the governing nonlinear partial differential equations to ordinary nonlinear differential equations and the Runge–Kutta method is used to obtain the ratio of linear to nonlinear frequencies. Frequency ratios obtained by the present theory are compared with the Mindlin plate theory results and Lo's 11-variable higher order plate theory results. It can be concluded that present modified plate theory predicts frequency ratios very accurately. Also, the benefit of significant simplification can be observed as compared with the Lo's higher order plate theory. The effects of initial stress and other factors on frequency ratio are investigated. © 2001 Published by Elsevier Science Ltd.

Keywords: Non-linear vibration; Modified plate theory

1. Introduction

Large amplitude vibration phenomenon of a plate plays an important role in works of obtaining design resonant-free structural components. So, the study of large amplitude vibration of plates has gained considerable importance in the recent years. The large amplitude vibration of plates are usually studied by using classical thin plate theory and Mindlin–Reissner type thick plate theory. A lot of references in the specialised monographs by Berger (1955), Chauhan and Ashwell (1969), Wu and Vinson (1969), Huang (1972), Chia and Parbhakrar (1978), Sathymoorthy (1979), Raju (1980), Reddy et al. (1981), Reddy and Chao (1982), Singh et al. (1991), Pillai and Rao (1991a,b) and Rao (1992) fully attest this statement.

However, an accurate determination of nonlinear vibration depends largely on the theory used to model a given structure. The above studies were based on the thin plate theory or Mindlin plate theory with a

* Corresponding author. Address: Banchiau P.O. Box 14-34, No. 132, Guo Ching Road, Banchiau City, Taipei County 220, Taiwan, ROC. Fax: +886-2-2958-7043.

PII: S0020-7683(00)00226-2

Nomenclature

C_{ij} stiffness coefficients of stress–strain relations
A_{ij}, D_{ij}, F_{ij}, H_{ij}, K_{ij} the stiffnesses of plate
σ_{ij}, $\overline{\sigma}_{ij}$ initial and perturbing stresses
u_s, $\overline{u}_s$ initial and perturbing displacements
$\overline{u}_i$ displacements of plate in x, y, z directions
u_x, u_y, w displacements of plate ($z = 0$) in x, y, z directions
φ_x, φ_y, φ_z rotations of plate in x, y directions
ξ_z higher-order shear deformation term
I_1, I_3, I_5, I_7 the inertia coefficients of plate
a, b dimensions of plate in x, y directions
h plate thickness
R aspect ratio, $R = a/b$
r thickness ratio, $r = a/h$
$W_{\max}$ nondimensional vibration amplitude, $W_{\max} = w/h$
N_{ij}, M_{ij} initial stress resultants
$\overline{P}_i$, ΔP_i the applied surface traction and perturbing surface traction
σ_{m} in-plane bending stress
σ_{n} in-plane normal stress
β ratio of bending stress to normal stress, $\beta = \sigma_{\mathrm{m}}/\sigma_{\mathrm{n}}$
K nondimensional buckling coefficient, $K = N_{xx}h^2r^2/\pi^2R^2D_{11}$
τ nondimensional time, $\tau = t(\pi^2D_{11}/\rho h^3a^2)^{1/2}$
ω_{l} nondimensional linear frequency
ω_{nl} nondimensional nonlinear frequency

shear correction factor introduced into the shear stress resultants, which in general violated the condition of surface traction. When the amplitude of vibrations are large enough compared to plate thickness, the thin plate and Mindlin plate theories do not adequately model the behaviour of structures. Higher order theory has a more realistic distribution of transverse shear strain and gives more accurate results than general first order plate theory. Many different higher order plate theories had been proposed by Whitney and Sun (1973), Lo et al. (1977a,b), Reddy (1984a,b) and Ahmed (1994) to plate analysis. Recently, Reddy and Phan (1985), Bhimaraddi and Stevens (1984) and Murth and Vellaichamy (1987) used a variational approach to derive a higher order theory in which a special displacement field is chosen to satisfy stress-free boundary conditions. A critical evaluation of new plate theories by Bert (1984) indicated that the theory of Lo and coworkers provides an accurate prediction in the behaviour of plates. However, this higher order plate theory with 11 variables is so complex that its utility is questionable. The third author and his colleague had derived the governing equations based on a higher order theory (Doong, 1987; Doong et al., 1987) and modified theory (Doong and Lee, 1991) for plates in a general state of nonuniform initial stress. The governing equations for plate deflection based on a modified higher order plate theory with seven kinematic variables. Firstly, assume the same form for the displacements as that of Lo et al. (1977a,b), then introduce the condition that the transverse shear stresses, $\overline{\sigma}_{xz}$ and $\overline{\sigma}_{yz}$, vanish on the top and bottom surfaces. These conditions are equivalent to the requirement that the corresponding strains are zero on these surfaces for plates. We have $\overline{\gamma}_{xz}(x, y, \pm h/2) = 0$ and $\overline{\gamma}_{yz}(x, y, \pm h/2) = 0$. In the previous works, only linear governing equations were derived to investigate natural frequencies and buckling loads of plates. It did not add shear correction factors but accurate results were obtained.

Using Mindlin plate theory (first order plate theory), Chen and Doong (1983a,b) studied the effects of an arbitrary initial stress state on the large amplitude vibration problems of plates. Owing to the complexity of nonlinear problems, little literature has been found to study large amplitude vibration problems based on a higher-order theory. Doong and Chen (1988) and Chen et al. (1994) used higher order theory to derive nonlinear governing equations of beams and plates, respectively, in which the large amplitude vibration problems were studied. In this paper, a modified higher order plate theory with seven variable is developed and the displacement field is chosen to satisfy stress-free boundary conditions. Nonlinear governing equations of motion are derived by using the average stress method on the basis of the von Karman assumption. The large amplitude vibration problems are studied by using the Galerkin method and Runge–Kutta method. The results of linear to nonlinear frequency ratio based on the modified higher order plate theory are compared with the Mindlin plate theory results and Lo's higher order plate theory results. Also, the effects of initial stress and other factors on frequency ratio are investigated.

2. Perturbed equations

We should consider a body in a state of nonuniform initial stress, which is in static equilibrium and is subjected to a time-varying incremental deformation. Following a technique described by Bolotin (1963) and Brunelle and Robertson (1974), the equations can be derived by using a perturbing technique:

$$(\sigma_{ij}\overline{u}_{s,j})_{,i} + [\overline{\sigma}_{ij}(\delta_{sj} + u_{s,j} + \overline{u}_{s,j})]_{,i} + \overline{F}_s + \Delta F_s = \rho\ddot{\overline{u}}_s, \tag{1}$$

$$\overline{P}_s + \Delta P_s = [\sigma_{ij}\overline{u}_{s,j} + \overline{\sigma}_{ij}(\delta_{js} + u_{s,j} + \overline{u}_{s,j})]n_i. \tag{2}$$

The n_i are the components of the unit normal given with respect to the spatial frame (a list of nomenclature is given). It is assumed that the initial displacement gradients are so small that the product $\overline{\sigma}_{ij}u_{s,j}$ can be neglected. For the plate theory of large deflection, the von Karman's assumptions are employed. Therefore, the perturbed displacement gradients are also so small that the terms $\overline{\sigma}_{ij}\overline{u}_{s,j}$ may be dropped except for $\overline{\sigma}_{ix}\overline{u}_{z,x}$ and $\overline{\sigma}_{iy}\overline{u}_{z,y}$. In order to give clarity to the integration procedure, it is useful to partially write out Eq. (1):

$$\partial(\sigma_{ij}\partial\overline{u}_x/\partial x_j)/\partial x_i + \partial\overline{\sigma}_{ix}/\partial x_i + \overline{F}_x + \Delta F_x = \rho\ddot{\overline{u}}_x, \tag{3}$$

$$\partial(\sigma_{ij}\partial\overline{u}_y/\partial x_j)/\partial x_i + \partial\overline{\sigma}_{iy}/\partial x_i + \overline{F}_y + \Delta F_y = \rho\ddot{\overline{u}}_y, \tag{4}$$

$$\partial(\sigma_{ij}\partial\overline{u}_z/\partial x_j)/\partial x_i + \partial\overline{\sigma}_{iz}/\partial x_i + \partial(\overline{\sigma}_{ix}\overline{u}_{z,x})/\partial x_i + \partial(\overline{\sigma}_{iy}\overline{u}_{z,y})/\partial x_i + \overline{F}_z + \Delta F_z = \rho\ddot{\overline{u}}_z. \tag{5}$$

3. Governing equations

The following seven-variable incremental displacement field that satisfied the stress-free boundary conditions (Doong and Lee, 1991) are assumed to be of the form

$$\overline{u}_x = u_x + z\left[\varphi_x - \tfrac{1}{2}\varphi_{z,x} - \tfrac{4}{3}(z/h)^2(w_{,x} + \varphi_x + \tfrac{1}{4}h^2\xi_{z,x})\right], \tag{6}$$

$$\overline{u}_y = u_y + z\left[\varphi_y - \tfrac{1}{2}\varphi_{z,y} - \tfrac{4}{3}(z/h)^2(w_{,y} + \varphi_y + \tfrac{1}{4}h^2\xi_{z,y})\right], \tag{7}$$

$$\overline{u}_z = w + z\varphi_z + z^2\xi_z. \tag{8}$$

The constitutive relations are given by

$$\begin{bmatrix} \overline{\sigma}_{xx} \\ \overline{\sigma}_{yy} \\ \overline{\sigma}_{zz} \\ \overline{\sigma}_{yz} \\ \overline{\sigma}_{zx} \\ \overline{\sigma}_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \overline{\varepsilon}_{xx} \\ \overline{\varepsilon}_{yy} \\ \overline{\varepsilon}_{zz} \\ \overline{\gamma}_{yz} \\ \overline{\gamma}_{zx} \\ \overline{\gamma}_{xy} \end{bmatrix}. \tag{9}$$

The von Karman's assumptions are employed. The displacements are infinitesimal. In the strain–displacement relations only those nonlinear terms which depend on $\overline{u}_{z,x}$, $\overline{u}_{z,y}$ are to be retained. All other nonlinear terms are to be neglected.

Kinematic relations are

$$\begin{aligned} \overline{\varepsilon}_{xx} &= \overline{u}_{x,x} + \tfrac{1}{2}\overline{u}_{z,x}^2, \\ \overline{\varepsilon}_{yy} &= \overline{u}_{y,y} + \tfrac{1}{2}\overline{u}_{z,y}^2, \\ \overline{\varepsilon}_{yy} &= \overline{u}_{z,z}, \\ \overline{\gamma}_{yz} &= \overline{u}_{y,z} + \overline{u}_{z,y}, \\ \overline{\gamma}_{zx} &= \overline{u}_{z,x} + \overline{u}_{x,z}, \\ \overline{\gamma}_{xy} &= \overline{u}_{x,y} + \overline{u}_{y,x} + \overline{u}_{z,x}\overline{u}_{z,y}. \end{aligned} \tag{10}$$

Thus, von Karman's theory differs from the linear theory only in retaining certain powers of the derivatives $\overline{u}_{z,x}$, $\overline{u}_{z,y}$ in the strain–displacement relationship.

Substituting the displacement field equations (6)–(8) into Eq. (10) and the constitution equation (9), the stress–displacement relations are found to be

$$\begin{aligned} \overline{\sigma}_{xx} = {} & C_{11}\left\{u_x + z\left[\varphi_x - \tfrac{1}{2}\varphi_{z,x} - \tfrac{4}{3}(z/h)^2(w_{,x} + \varphi_x + \tfrac{1}{4}h^2\xi_{z,x})\right]\right\}_{,x} \\ & + C_{12}\left\{u_y + z\left[\varphi_y - \tfrac{1}{2}\varphi_{z,y} - \tfrac{4}{3}(z/h)^2(w_{,y} + \varphi_y + \tfrac{1}{4}h^2\xi_{z,y})\right]\right\}_{,y} \\ & + C_{13}[w + z\varphi_z + z^2\xi_z]_{,z} + C_{11}(w + z\varphi_z + z^2\xi_z)_{,x}^2/2 + C_{12}(w + z\varphi_z + z^2\xi_z)_{,y}^2/2, \end{aligned} \tag{11}$$

$$\begin{aligned} \overline{\sigma}_{yy} = {} & C_{12}\left\{u_x + z\left[\varphi_x - \tfrac{1}{2}\varphi_{z,x} - \tfrac{4}{3}(z/h)^2(w_{,x} + \varphi_x + \tfrac{1}{4}h^2\xi_{z,x})\right]\right\}_{,x} \\ & + C_{22}\left\{u_y + z\left[\varphi_y - \tfrac{1}{2}\varphi_{z,y} - \tfrac{4}{3}(z/h)^2(w_{,y} + \varphi_y + \tfrac{1}{4}h^2\xi_{z,y})\right]\right\}_{,y} \\ & + C_{23}[w + z\varphi_z + z^2\xi_z]_{,z} + C_{12}(w + z\varphi_z + z^2\xi_z)_{,x}^2/2 + C_{22}(w + z\varphi_z + z^2\xi_z)_{,y}^2/2, \end{aligned} \tag{12}$$

$$\begin{aligned} \overline{\sigma}_{zz} = {} & C_{13}\left\{u_x + z\left[\varphi_x - \tfrac{1}{2}\varphi_{z,x} - \tfrac{4}{3}(z/h)^2\left(w_{,x} + \varphi_x + \tfrac{1}{4}h^2\xi_{z,x}\right)\right]\right\}_{,x} \\ & + C_{23}\left\{u_y + z\left[\varphi_y - \tfrac{1}{2}\varphi_{z,y} - \tfrac{4}{3}(z/h)^2\left(w_{,y} + \varphi_y + \tfrac{1}{4}h^2\xi_{z,y}\right)\right]\right\}_{,y} \\ & + C_{33}[w + z\varphi_z + z^2\xi_z]_{,z} + C_{13}(w + z\varphi_z + z^2\xi_z)_{,x}^2/2 + C_{23}(w + z\varphi_z + z^2\xi_z)_{,y}^2/2, \end{aligned} \tag{13}$$

$$\overline{\sigma}_{yz} = C_{44}\left(\left\{u_y + z\left[\varphi_y - \tfrac{1}{2}\varphi_{z,y} - \tfrac{4}{3}(z/h)^2\left(w_{,y} + \varphi_y + \tfrac{1}{4}h^2\xi_{z,y}\right)\right]\right\}_{,z} + [w + z\varphi_z + z^2\xi_z]_{,y}\right), \tag{14}$$

$$\overline{\sigma}_{xz} = C_{55}\left(\left\{u_x + z\left[\varphi_x - \tfrac{1}{2}\varphi_{z,x} - \tfrac{4}{3}(z/h)^2\left(w_{,x} + \varphi_x + \tfrac{1}{4}h^2\xi_{z,x}\right)\right]\right\}_{,z} + [w + z\varphi_z + z^2\xi_z]_{,x}\right), \tag{15}$$

$$\overline{\sigma}_{xy} = C_{66}\Bigg(\Big\{u_x + z\Big[\varphi_x - \tfrac{1}{2}\varphi_{z,x} - \tfrac{4}{3}(z/h)^2\big(w_{,x} + \varphi_x + \tfrac{1}{4}h^2\xi_{z,x}\big)\Big]\Big\}_{,y} + \Big\{u_y + z\Big[\varphi_y - \tfrac{1}{2}\varphi_{z,y} - \tfrac{4}{3}(z/h)^2\big(w_{,y} + \varphi_y + \tfrac{1}{4}h^2\xi_{z,y}\big)\Big]\Big\}_{,x} + [w + z\varphi_z + z^2\xi_z]_{,x}[w + z\varphi_z + z^2\xi_z]_{,y}\Bigg). \tag{16}$$

For subsequent use in the equations of motion, the following initial stress resultants and material parameters are defined:

$$\begin{aligned}
&(A_{ij}, D_{ij}, F_{ij}, H_{ij}, K_{ij}) = \int C_{ij}(1, z^2, z^4, z^6, z^8)\,dz \quad (i,j = 1,2,3,4,5,6),\\
&(N_{ij}, M_{ij}, M^*_{ij}, P_{ij}, P^*_{ij}, R_{ij}, R^*_{ij}) = \int \sigma_{ij}(1, z, z^2, z^3, z^4, z^5, z^6)\,dz \quad (i,j = x,y,z),\\
&(I_1, I_3, I_5, I_7) = \int \rho(1, z^2, z^4, z^6)\,dz,
\end{aligned} \tag{17}$$

where all the integrals are through the thickness of the plate from $-h/2$ to $h/2$. The seven governing equations can be obtained by substituting Eqs. (11)–(16) into Eqs. (3)–(5) and integrating equations. These governing equations are as follows:

$$(Q_1 + L_1 + L_{11})_{,x} + (Q_2 + L_{21})_{,y} + (R_1 + R_5 + R_{17})_{,x} + (S_1 + S_6 + S_{17})_{,y} + f_x = I_1\ddot{u}_x - \tfrac{1}{2}I_3\ddot{\varphi}_{z,x}, \tag{18}$$

$$(Q_2 + L_{21})_{,x} + (Q_3 + L_4 + L_{14})_{,y} + (R_9 + R_{13} + R_{21})_{,x} + (S_9 + S_{13} + S_{21})_{,y} + f_y = I_1\ddot{u}_y - \tfrac{1}{2}I_3\ddot{\varphi}_{z,y}, \tag{19}$$

$$\begin{aligned}
&(Q_4 + L_{26} + L_{29})_{,x} + (Q_5 + L_{32} + L_{35})_{,y} + (R_{30} + R_{33} + R_{25})_{,x} + (S_{30} + S_{33} + S_{25})_{,y} + f_z\\
&\quad + \left(\frac{4}{3h^2}\right)\Big\{\Big[(Q_{18} + L_8 + L_{18})_{,xx} + 2(Q_{19} + L_{25})_{,xy} + (R_4 + R_8 + R_{20})_{,xx} + (S_4 + S_8 + S_{20})_{x,y}\\
&\quad - 3Q_{15,x} + (Q_{20} + L_{10} + L_{20})_{,yy} + (R_{12} + R_{16} + R_{24})_{,xy} + (S_{12} + S_{16} + S_{24})_{,yy} - 3Q_{16,y} + q_{x,x} + q_{y,y}\Big]\Big\}\\
&\quad = I_1\ddot{w} + I_3\ddot{\xi}_z + \left(\frac{4}{3h^2}\right)\overline{I}_5(\ddot{\varphi}_{x,x} + \ddot{\varphi}_{y,y}) - \left(\frac{4}{3h^2}\right)^2 I_7\left[\ddot{w}_{,xx} + \frac{h^2}{4}\ddot{\xi}_{z,xx} + \ddot{w}_{,yy} + \frac{h^2}{4}\ddot{\xi}_{z,yy}\right],
\end{aligned} \tag{20}$$

$$\begin{aligned}
&(Q_6 + L_7 + L_{17})_{,x} + (Q_7 + L_{24})_{,y} + (R_2 + R_6 + R_{18})_{,x} + (S_2 + S_6 + S_{18})_{,y} - Q_4 + m_x\\
&\quad - \left(\frac{4}{3h^2}\right)[(Q_{18} + L_8 + L_{18})_{,x} + (Q_{19} + L_{25})_{,y} + (R_4 + R_8 + R_{20})_{,x} + (S_4 + S_8 + S_{20})_{,y} - 3Q_{15} + q_x]\\
&\quad = \overline{I}_3\ddot{\varphi}_x - \left(\frac{4}{3h^2}\right)\overline{I}_5\left[\ddot{w}_{,x} + \ddot{\varphi}_x + \frac{h^2}{4}\ddot{\xi}_{z,x}\right],
\end{aligned} \tag{21}$$

$$\begin{aligned}
&(Q_7 + L_{24})_{,x} + (Q_8 + L_9 + L_{19})_{,y} + (R_{10} + R_{14} + R_{22})_{,x} + (S_{10} + S_{14} + S_{22})_{,y} - Q_5 + m_y\\
&\quad - \left(\frac{4}{3h^2}\right)[(Q_{19} + L_{25})_{,x} + (Q_{20} + L_{10} + L_{20})_{,y} + (R_{12} + R_{16} + R_{24})_{,x} + (S_{12} + S_{16} + S_{24})_{,y}\\
&\quad - 3Q_{16} + q_y] = \overline{I}_3\ddot{\varphi}_y - \left(\frac{4}{3h^2}\right)\overline{I}_5\left[\ddot{w}_{,y} + \ddot{\varphi}_y + \frac{h^2}{4}\ddot{\xi}_{z,y}\right],
\end{aligned} \tag{22}$$

$$\begin{aligned}
&(L_{27} + L_{30})_{,x} + (L_{33} + L_{36})_{,y} + (R_{31} + R_{34} + R_{36})_{,x} + (S_{31} + S_{34} + S_{36})_{,y} - Q_{11} - Q_{26} + m_z + \tfrac{1}{2}[(Q_{12}\\
&\quad + L_2 + L_{12})_{,xx} + 2(Q_{13} + L_{22})_{,xy} + (R_3 + R_7 + R_{19})_{,xx} + (S_3 + S_7 + S_{19})_{,xy} + (Q_{14} + L_5 + L_{15})_{,yy}\\
&\quad + (R_{11} + R_{15} + R_{23})_{,xy} + (S_{11} + S_{15} + S_{23})_{,yy} + h_{x,x} + h_{y,y}] - L_{40}\\
&\quad = I_3\ddot{\varphi}_z + \tfrac{1}{2}I_3(\ddot{u}_{x,x} + \ddot{u}_{y,y}) - \tfrac{1}{4}I_5(\ddot{\varphi}_{z,xx} + \ddot{\varphi}_{z,yy}),
\end{aligned} \tag{23}$$

$$(L_{28}+L_{31})_{,x}+(L_{34}+L_{37})_{,y}+(R_{32}+R_{35}+R_{27})_{,x}+(S_{32}+S_{35}+S_{27})_{,y}+h_z-\tfrac{1}{3}[(Q_{18}+L_8+L_{18})_{,xx}$$
$$+2(Q_{19}+L_{25})_{,xy}+(R_4+R_8+R_{20})_{,xx}+(S_4+S_8+S_{20})_{,xy}+(Q_{20}+L_{10}+L_{20})_{,yy}$$
$$+(R_{12}+R_{16}+R_{24})_{,xy}+(S_{12}+S_{16}+S_{24})_{,yy}+q_{x,x}+q_{y,y}]-2Q_{17}-L_{41}$$
$$=I_3\ddot{w}+I_5\ddot{\xi}_z+\tfrac{1}{3}\overline{I}_5(\ddot{\varphi}_{x,x}+\ddot{\varphi}_{y,y})-\left(\frac{4}{9h^2}\right)I_7\left[\ddot{w}_{,xx}+\frac{h^2}{4}\ddot{\xi}_{z,xx}+\ddot{w}_{,yy}+\frac{h^2}{4}\ddot{\xi}_{z,yy}\right]. \tag{24}$$

The coefficients of above equations are given in Appendix A.

Before giving the boundary conditions on the plate, in terms of normal and tangential coordinates, it is convenient to define the following traction resultants where all integrals are from $-h/2$ to $h/2$:

$$\Delta F_{nn}=\int\Delta P_n\,\mathrm{d}z,\qquad \Delta\overline{F}_{nn}=\int\overline{P}_n\,\mathrm{d}z,$$
$$\Delta F_{nt}=\int\Delta P_t\,\mathrm{d}z,\qquad \Delta\overline{F}_{nt}=\int\overline{P}_t\,\mathrm{d}z,$$
$$\Delta F_{nz}=\int\Delta P_z\,\mathrm{d}z,\qquad \Delta\overline{F}_{nz}=\int\overline{P}_z\,\mathrm{d}z,$$
$$\Delta M_{nn}=\int\Delta P_n z\,\mathrm{d}z,\qquad \Delta\overline{M}_{nn}=\int\overline{P}_n z\,\mathrm{d}z,$$
$$\Delta M_{nt}=\int\Delta P_t z\,\mathrm{d}z,\qquad \Delta\overline{M}_{nt}=\int\overline{P}_t z\,\mathrm{d}z,$$
$$\Delta M_{nz}=\int\Delta P_z z\,\mathrm{d}z,\qquad \Delta\overline{M}_{nz}=\int\overline{P}_z z\,\mathrm{d}z,$$
$$\Delta M^*_{nz}=\int\Delta P_z z^2\,\mathrm{d}z,\qquad \Delta\overline{M}^*_{nz}=\int\overline{P}_z z^2\,\mathrm{d}z,$$
$$(\hat{N}_{ij},\hat{M}_{ij},\hat{M}^*_{ij},\hat{P}_{ij},\hat{P}^*_{ij})=\int\overline{\sigma}_{ij}(1,z,z^2,z^3,z^4)\,\mathrm{d}z,\quad (i,j=x,y,z).$$

If Eq. (2) is phrased in terms of stresses normal and tangent to the edge then $n_n=1$, $n_t=0$ and $n_z=0$, on the edge. Thus,

$$\overline{P}_n+\Delta P_n=\sigma_{nn}\overline{u}_{n,n}+\sigma_{nt}\overline{u}_{n,t}+\sigma_{nz}\overline{u}_{n,z}+\overline{\sigma}_{nn}, \tag{25}$$

$$\overline{P}_t+\Delta P_t=\sigma_{nn}\overline{u}_{t,n}+\sigma_{nt}\overline{u}_{t,t}+\sigma_{nz}\overline{u}_{t,z}+\overline{\sigma}_{nt}, \tag{26}$$

$$\overline{P}_z+\Delta P_z=\sigma_{nn}\overline{u}_{z,n}+\sigma_{nt}\overline{u}_{z,t}+\sigma_{nz}\overline{u}_{z,z}+\overline{\sigma}_{nz}+\overline{\sigma}_{nn}\overline{u}_{z,n}+\overline{\sigma}_{nt}\overline{u}_{z,t}. \tag{27}$$

The subscripts n and t denote the normal and tangential directions of the plate's edge, respectively. The terms containing the initial stresses account for the change in the initial boundary conditions due to the incremental deformation. The boundary traction conditions are

$$
\begin{aligned}
\overline{F}_{nn} + \Delta F_{nn} &= N_{nn}u_{n,n} + M_{nn}\varphi_{n,n} - \frac{1}{2}M_{nn}\varphi_{z,nn} - \frac{4}{3h^2}P_{nn}\left(w_{,nn} + \varphi_{n,n} + \frac{h^2}{4}\xi_{z,nn}\right) \\
&\quad + N_{nt}u_{n,t} + M_{nt}\varphi_{n,t} - \frac{1}{2}M_{nt}\varphi_{z,nt} - \frac{4}{3h^2}P_{nt}\left(w_{,nt} + \varphi_{n,t} + \frac{h^2}{4}\xi_{z,nt}\right) \\
&\quad + N_{nz}\varphi_n - \frac{1}{2}N_{nz}\varphi_{z,n} - \frac{4}{h^2}M^*_{nz}\left(w_{,n} + \varphi_n + \frac{h^2}{4}\xi_{z,n}\right) + \hat{N}_{nn}, \\
\overline{F}_{nt} + \Delta F_{nt} &= N_{nn}u_{t,n} + M_{nn}\varphi_{t,n} - \frac{1}{2}M_{nn}\varphi_{z,tn} - \frac{4}{3h^2}P_{nn}\left(w_{,tn} + \varphi_{t,n} + \frac{h^2}{4}\xi_{z,tn}\right) \\
&\quad + N_{nt}u_{t,t} + M_{nt}\varphi_{t,t} - \frac{1}{2}M_{nt}\varphi_{z,tt} - \frac{4}{3h^2}P_{nt}\left(w_{,tt} + \varphi_{t,t} + \frac{h^2}{4}\xi_{z,tt}\right) \\
&\quad + N_{nz}\varphi_t - \frac{1}{2}N_{nz}\varphi_{z,t} - \frac{4}{h^2}M^*_{nz}\left(w_{,t} + \varphi_t + \frac{h^2}{4}\xi_{z,t}\right) + \hat{N}_{nt}, \\
\overline{F}_{nz} + \Delta F_{nz} &= N_{nn}w_{,n} + M_{nn}\varphi_{z,n} + M^*_{nn}\xi_{z,n} + N_{nt}w_{,t} + M_{nt}\varphi_{z,t} + M^*_{nt}\xi_{z,t} + N_{nz}\varphi_z \\
&\quad + 2M_{nz}\xi_z + \hat{N}_{nz} + \hat{N}_{nn}w_{,n} + \hat{M}_{nn}\varphi_{z,n} + \hat{M}^*_{nn}\xi_{z,n} + \hat{N}_{nt}w_{,t} + \hat{M}_{nt}\varphi_{z,t} + \hat{M}^*_{nt}\xi_{z,t}, \\
\overline{M}_{nn} + \Delta M_{nn} &= M_{nn}u_{n,n} + M^*_{nn}\varphi_{n,n} - \frac{1}{2}M^*_{nn}\varphi_{z,nn} - \frac{4}{3h^2}P^*_{nn}\left(w_{,nn} + \varphi_{n,n} + \frac{h^2}{4}\xi_{z,nn}\right) + M_{nt}u_{n,t} \\
&\quad + M^*_{nt}\varphi_{n,t} - \frac{1}{2}M^*_{nt}\varphi_{z,nt} - \frac{4}{3h^2}P^*_{nt}\left(w_{,nt} + \varphi_{n,t} + \frac{h^2}{4}\xi_{z,nt}\right) \\
&\quad + M_{nz}\varphi_n - \frac{1}{2}M_{nz}\varphi_{z,n} - \frac{4}{h^2}P_{nz}\left(w_{,n} + \varphi_n + \frac{h^2}{4}\xi_{z,n}\right) + \hat{M}_{nn}, \\
\overline{M}_{nt} + \Delta M_{nt} &= M_{nn}u_{t,n} + M^*_{nn}\varphi_{t,n} - \frac{1}{2}M^*_{nn}\varphi_{z,tn} - \frac{4}{3h^2}P^*_{nn}\left(w_{,tn} + \varphi_{t,n} + \frac{h^2}{4}\xi_{z,tn}\right) \\
&\quad + M_{nt}u_{t,t} + M^*_{nt}\varphi_{t,t} - \frac{1}{2}M^*_{nt}\varphi_{z,tt} - \frac{4}{3h^2}P^*_{nt}\left(w_{,tt} + \varphi_{t,t} + \frac{h^2}{4}\xi_{z,tt}\right) \\
&\quad + M_{nz}\varphi_t - \frac{1}{2}M_{nz}\varphi_{z,t} - \frac{4}{h^2}P_{nz}\left(w_{,t} + \varphi_t + \frac{h^2}{4}\xi_{z,t}\right) + \hat{M}_{nt}, \\
\overline{M}_{nz} + \Delta M_{nz} &= M_{nn}w_{,n} + M^*_{nn}\varphi_{z,n} + P_{nn}\xi_{z,n} + M_{nt}w_{,t} + M^*_{nt}\varphi_{z,t} + P_{nt}\xi_{z,t} + M_{nz}\varphi_z + 2M^*_{nz}\xi_z \\
&\quad + \hat{M}_{nz} + \hat{M}_{nn}w_{,n} + \hat{M}^*_{nn}\varphi_{z,n} + \hat{P}_{nn}\xi_{z,n} + \hat{M}_{nt}w_{,t} + \hat{M}^*_{nt}\varphi_{z,t} + \hat{P}_{nt}\xi_{z,t}, \\
\overline{M}^*_{nz} + \Delta M^*_{nz} &= M^*_{nn}w_{,n} + P_{nn}\varphi_{z,n} + P^*_{nn}\xi_{z,n} + M^*_{nt}w_{,t} + P_{nt}\varphi_{z,t} + P^*_{nt}\xi_{z,t} + M^*_{nz}\varphi_z + 2P_{nz}\xi_z \\
&\quad + \hat{M}^*_{nz} + \hat{M}^*_{nn}w_{,n} + \hat{P}_{nn}\varphi_{z,n} + \hat{P}^*_{nn}\xi_{z,n} + \hat{M}^*_{nt}w_{,t} + \hat{P}_{nt}\varphi_{z,t} + \hat{P}^*_{nt}\xi_{z,t}.
\end{aligned}
\tag{28}
$$

Alternative displacement boundary conditions are

$$u_n = u_{nn}, \quad u_t = u_{nt}, \quad w = w_{nz}, \quad \varphi_n = \varphi_{nn}, \quad \varphi_t = \varphi_{nt}, \quad \varphi_z = \varphi_{nz}, \quad \xi_z = \xi_{nz},$$

where the quantities on the right-hand side are prescribed. If a rectangular plate was being considered, the boundary conditions would be rephrased in x, y coordinates.

4. Example problem

Here a simply supported plate of uniform thickness h in a state of initial stress is to be considered. The state of initial stress is

$$\sigma_{xx} = \sigma_{\rm n} + 2z\sigma_{\rm m}/h, \tag{29}$$

where $\sigma_{\rm m}$ and $\sigma_{\rm n}$ are taken to be constants and other initial stresses assumed to be zero. It is comprised of a tensile (compressive) $\sigma_{\rm n}$ plus a bending stress $\sigma_{\rm m}$. The only nonzero initial stress resultants are

$$N_{xx} = h\sigma_{\rm n}, \qquad M_{xx} = h^2\sigma_{\rm m}/6, \qquad M^*_{xx} = h^3\sigma_{\rm n}/12, \qquad P_{xx} = h^4\sigma_{\rm m}/40, \qquad P^*_{xx} = h^5\sigma_{\rm n}/80.$$

Lateral loads and body forces are taken to be zero:

$$f_x, f_y, f_z, h_x, h_y, h_z, m_x, m_y, m_z, q_x, q_y = 0.$$

For the simply supported plate, the boundary conditions are, on $x = 0$ and $x = a$ edges

$$\begin{aligned} &u_y = 0, \quad \varphi_y = 0, \quad w = 0, \quad \varphi_z = 0, \quad \xi_z = 0, \\ &\overline{F}_{xx} + \Delta F_{xx} = N_{xx}u_{x,x} + M_{xx}\varphi_{x,x} - \frac{1}{2}M_{xx}\varphi_{z,xx} - \frac{4}{3h^2}P_{xx}\left(w_{,xx} + \varphi_{x,x} + \frac{h^2}{4}\xi_{z,xx}\right) + \hat{N}_{xx} = 0, \\ &M_{xx} + \Delta M_{xx} = M_{xx}u_{x,x} + M^*_{xx}\varphi_{x,x} - \frac{1}{2}M^*_{xx}\varphi_{z,xx} - \frac{4}{3h^2}P^*_{xx}\left(w_{,xx} + \varphi_{x,x} + \frac{h^2}{4}\xi_{z,xx}\right) + \hat{M}_{xx} = 0 \end{aligned} \tag{30}$$

on the $x = 0$ and $x = a$ edges

$$\begin{aligned} &u_x = 0, \quad \varphi_x = 0, \quad w = 0, \quad \varphi_z = 0, \quad \xi_z = 0, \\ &\overline{F}_{yy} + \Delta F_{yy} = N_{yy}u_{y,y} + M_{yy}\varphi_{y,y} - \frac{1}{2}M_{yy}\varphi_{z,yy} - \frac{4}{3h^2}P_{yy}\left(w_{,yy} + \varphi_{y,y} + \frac{h^2}{4}\xi_{z,yy}\right) + \hat{N}_{yy} = 0, \\ &\overline{M}_{yy} + \Delta M_{yy} = M_{yy}u_{y,y} + M^*_{yy}\varphi_{y,y} - \frac{1}{2}M^*_{yy}\varphi_{z,yy} - \frac{4}{3h^2}P^*_{yy}\left(w_{,yy} + \varphi_{y,y} + \frac{h^2}{4}\xi_{z,yy}\right) + \hat{M}_{yy} = 0. \end{aligned} \tag{31}$$

The following one term fundamental mode shapes satisfy the boundary conditions equations (30) and (31)

$$\begin{aligned} u_x &= hU(t)\cos(\pi x/a)\sin(\pi y/b), \\ u_y &= hV(t)\sin(\pi x/a)\cos(\pi y/b), \\ w &= hW(t)\sin(\pi x/a)\sin(\pi y/b), \\ \varphi_x &= \Psi_x(t)\cos(\pi x/a)\sin(\pi y/b), \\ \varphi_y &= \Psi_y(t)\sin(\pi x/a)\cos(\pi y/b), \\ \varphi_z &= \Psi_z(t)\sin(\pi x/a)\sin(\pi y/b), \\ \xi_z &= \zeta_z(t)/h\sin(\pi x/a)\sin(\pi y/b). \end{aligned} \tag{32}$$

Then, by substituting the assumed displacement fields of Eq. (32) into the equations of motion (18)–(24) and solving by Galerkin method, one obtains

$$C_{1,1}U + C_{1,2}V + C_{1,4}\Psi_x + C_{1,6}\Psi_z + N_1W^2 + N_2\Psi_z^2 + N_3\zeta_z^2 + N_5W\zeta_z = R_{1,1}\ddot{U} + R_{1,6}\ddot{\Psi}_z, \tag{33}$$

$$C_{1,2}U + C_{2,2}V + C_{2,5}\Psi_y + C_{2,6}\Psi_z + N_8W^2 + N_9\Psi_z^2 + N_{10}\zeta_z^2 + N_{12}W\zeta_z = R_{2,2}\ddot{V} + R_{2,6}\ddot{\Psi}_z, \tag{34}$$

$$\begin{aligned} &C_{3,3}W + C_{3,4}\Psi_x + C_{3,5}\Psi_y + C_{3,6}\Psi_z + C_{3,7}\zeta_z + N_{15}UW + N_{16}U\zeta_z + N_{19}VW + N_{20}V\zeta_z + N_{28}W\Psi_z \\ &\quad + N_{16}(\Psi_x + \Psi_y)\Psi_z + N_{29}\Psi_z\zeta_z + N_{17}\Psi_z\Phi_z + N_{21}\Psi_z\Phi_y + N_{36}\Psi_z\zeta_z + N_{23}W^3 \\ &\quad + 3(N_{24}W^2\zeta_z + N_{24}\Psi_z^2W + N_{25}\Psi_z^2 + N_{25}\zeta_z^2W) + N_{26}\zeta_z^3 \\ &\quad = R_{3,3}\ddot{W} + R_{3,4}\ddot{\Psi}_x + R_{3,5}\ddot{\Psi}_y + R_{3,7}\ddot{\zeta}_z, \end{aligned} \tag{35}$$

$$C_{1,4}U + C_{3,4}W + C_{4,4}\Psi_x + C_{4,5}\Psi_y + C_{4,7}\zeta_z + N_{40}W\Psi_z + N_{42}\Psi_z\zeta_z = R_{3,4}\ddot{W} + R_{4,4}\ddot{\Psi}_x + R_{4,7}\ddot{\zeta}_z, \tag{36}$$

$$C_{2,5}V + C_{3,5}W + C_{4,5}\Psi_x + C_{5,5}\Psi_y + C_{5,7}\zeta_z + N_{41}W\Psi_z + N_{43}\Psi_z\zeta_z = R_{3,5}\ddot{W} + R_{5,5}\ddot{\Psi}_y + R_{5,7}\ddot{\zeta}_z, \tag{37}$$

$$\begin{aligned} &C_{1,6}U + C_{2,6}V + C_{3,6}W + C_{6,6}\Psi_z + C_{6,7}\zeta_z + N_{16}U\Psi_z + N_{20}V\Psi_z + 2(N_{28} + N_{31})W^2 \\ &\quad + (6N_{28} + 4N_{31})W\zeta_z + 3N_{29}\Psi_z^2 + 3N_{24}W^2\Psi_z + N_{25}\Psi_z^3 + 3N_{26}\zeta_z^2\Psi_z + 6N_{25}W\Psi_z\zeta_z + N_{37}W^2 \\ &\quad + N_{38}\Psi_z^2 + N_{39}\zeta_z^2 \\ &\quad = R_{1,6}\ddot{U} + R_{2,6}\ddot{V} + R_{6,6}\ddot{\Psi}_z, \end{aligned} \tag{38}$$

$$\begin{aligned} &C_{3,7}W + C_{4,7}\Psi_z + C_{5,7}\Psi_y + C_{7,7}\zeta_z + N_{16}UW + N_{17}U\zeta_z + N_{20}VW + N_{21}V\zeta_z + (3N_{20} + 4N_{32})W\Psi_z \\ &\quad + (N_{17} + 2N_{36})\Psi_x\Psi_z + (N_{17} + 2N_{40})\Psi_y\Psi_z + (3N_{30} + 4N_{33})\Psi_z\zeta_z + N_{24}W^3 \\ &\quad + 3(N_{25}W^2\zeta_z + N_{25}W\Psi_z^2 + N_{26}\Psi_z^2\zeta_z + N_{26}\zeta_z^2W) + N_{27}\zeta_z^3 \\ &\quad = R_{3,7}\ddot{W} + R_{4,7}\ddot{\Psi}_x + R_{5,7}\ddot{\Psi}_y + R_{7,7}\ddot{\zeta}_z. \end{aligned} \tag{39}$$

The coefficients of above equations are given in Appendix B.

The equations are integrated by using the fourth order Runge–Kutta method with the nondimension time interval $\Delta\tau$ taken as 0.001 to obtain reasonably accurate results. In each case, the initial conditions are chosen as

$$U(0) = V(0) = \Psi_x(0) = \Psi_y(0) = \Psi_z(0) = \zeta_z(0), \qquad W(0) = W_{\max},$$

$$U_{,t}(0) = V_{,t}(0) = \Psi_{x,t}(0) = \Psi_{y,t}(0) = \Psi_{z,t}(0) = \zeta_{z,t}(0) = W_{,t}(0) = 0.$$

The initial in-plane compressive (tensile) stress is contained in the buckling coefficient K. If K is positive, then the stress is tensile. The initial in-plane bending stress contained in β, when $\beta = 0$ and $K = 0$, there is no initial stress. $W_{\max}$ varies from 0.2 to 1.0 to show the different characteristics between small amplitude vibration and large amplitude vibration. The nonlinear frequency for one full cycle is measured as $T_{\rm nl}$, and the nonlinear frequency is computed as $\omega_{\rm nl} = 1/T_{\rm nl}$. The linear frequency $\omega_{\rm l}$ can be calculated by neglecting the nonlinear terms in Eqs. (27)–(33).

5. Results and discussion

There are so many parameters that can be varied, it is difficult to present results for all cases. From the numerous problems solved, only a few typical cases will be selected for discussion. The large amplitude vibration behaviour is described by the linear to nonlinear ratio of vibration frequency ($\omega_{\rm l}/\omega_{\rm nl}$). Table 1 gives the frequency ratios for various amplitude ratios, the isotropic plate with no initial stresses are the first to be considered and the frequency ratios based on present seven-variable higher order theory are compared with Sathymoorthy's results (1979), Doong's previous Mindlin plate theory results and 11-variable higher order plate theory results (Chen et al., 1994). It can be observed that the frequency ratios of small amplitude calculated by the seven-variable higher order theory matches with the other theory results very well. However, the difference among them increases as the vibration amplitude increases. The comparison between seven variable and 11-variable plate theory results are shown in Figs. 1–3. From Fig. 1, it can be seen that the results calculated with modified higher order plate model indicate the same decreasing trend as with previous higher order theory model. Also, the previous higher order results always have a smaller value than modified higher order theory results. At small amplitude the frequency ratios have very nearly the

Table 1
ω_l/ω_{nl} for isotropic square plate ($a/b = 1$, $a/h = 10$, $v = 0.3$, $\beta = 0$)

	$W_{max} = 0.2$	$W_{max} = 0.4$	$W_{max} = 0.6$	$W_{max} = 0.8$	$W_{max} = 1.0$
A[a]	0.973	0.906	0.820	0.733	0.655
B[b]	0.968	0.885	0.781	0.688	0.603
H[c]	0.966	0.878	0.767	0.699	0.581
M[d]	0.967	0.879	0.780	0.683	0.594

[a] Sathymoorthy's results (Sathymoorthy, 1979).
[b] Doong's first order theory results.
[c] Previous higher order theory results (Chen et al., 1994).
[d] Present higher order theory results.

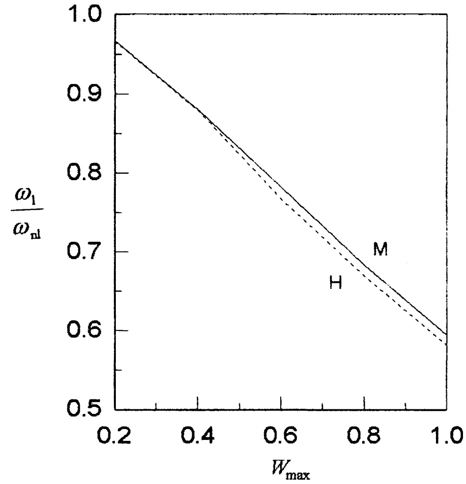

Fig. 1. Comparison of present results with previous higher order theory ($a/b = 1$, $a/h = 10$, $\beta = 0$, $v = 0.3$, $K = 0$, H: previous higher order theory results; M: present higher order theory results).

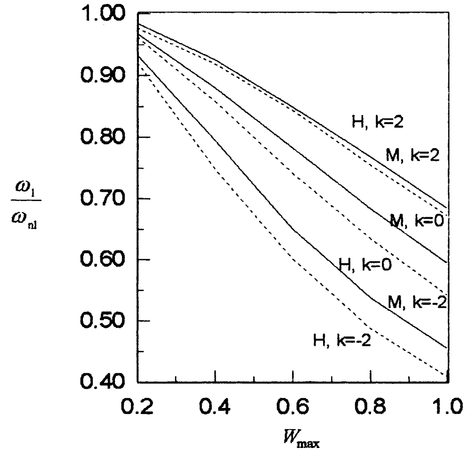

Fig. 2. Frequency ratio versus vibration amplitude for various values of K ($a/b = 1$, $a/h = 10$, $\beta = 0$, $v = 0.3$, H: 11-variable higher order theory results; M: present seven-variable higher order theory results).

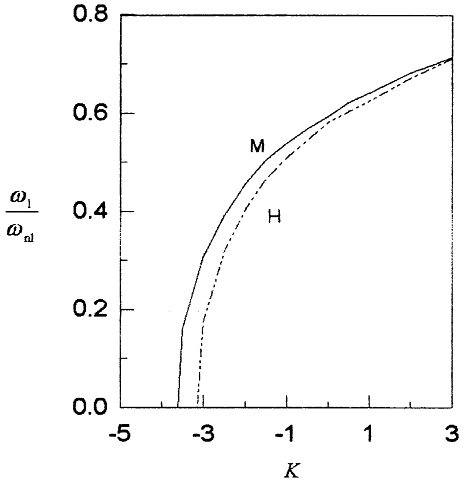

Fig. 3. Frequency ratio versus buckling coefficient ($a/b = 1$, $a/h = 10$, $\beta = 0$, $v = 0.3$, $W_{max} = 1$, H: 11-variable higher order theory results; M: present seven-variable higher order theory results).

same data for both models while at large amplitude the little difference between them can be observed. Both seven variable and 11-variable plate theories have almost the same accuracy. However, the benefit of present theory, simplification over previous theory, makes this theory suitable to treat plate problems.

Table 2 present the effect of the nondimensional thickness ratio a/h on the frequency ratio. It can be found that the frequency ratio increases with increasing thickness ratio and the difference between two higher order theories are more significant at low thickness ratio, i.e. the stress-free boundary conditions should be considered for the very thick plate condition. Tables 3 and 4 depict the difference of two theory results for various of aspect ratio and Poisson's ratios, respectively. It can be seen that the present seven-variable results indicate similar frequency ratios as 11-variable higher order theory results. The differences

Table 2
ω_l/ω_{nl} for isotropic square plate ($a/b = 1$, $W_{max} = 1$, $v = 0.3$, $\beta = 0$)

	$a/h = 5$	$a/h = 10$	$a/h = 20$	$a/h = 30$	$a/h = 40$	$a/h = 50$
F[a]	0.567	0.605	0.612	0.613	0.614	0.615
H[b]	0.543	0.581	0.597	0.613	0.618	0.619
M[c]	0.562	0.594	0.609	0.611	0.611	0.611

[a] First order theory results.
[b] Eleven-variable higher order theory results.
[c] Present seven-variable higher order theory results.

Table 3
ω_l/ω_{nl} for various values of aspect ratio ($a/h = 10$, $W_{max} = 1$, $v = 0.3$, $\beta = 0$)

	$a/b = 0.2$	$a/b = 0.4$	$a/b = 0.6$	$a/b = 0.8$	$a/b = 1.0$
F[a]	0.4622	0.4966	0.5473	0.5918	0.6026
H[b]	0.4311	0.4682	0.5240	0.5682	0.5811
M[c]	0.4554	0.4821	0.5371	0.5914	0.5938

[a] First order theory results.
[b] Eleven-variable higher order theory results.
[c] Present seven-variable higher order theory results.

Table 4
ω_l/ω_{nl} for various values of Poisson's ratio ($a/b=1$, $a/h=10$, $\beta=0$)

ν		$W_{max}=0.2$	$W_{max}=0.4$	$W_{max}=0.6$	$W_{max}=0.8$	$W_{max}=1.0$
0.1	F[a]	0.9711	0.8930	0.7936	0.7032	0.6149
	H[b]	0.9710	0.8927	0.7923	0.7007	0.6135
	M[c]	0.9628	0.8925	0.7924	0.6947	0.6115
0.2	F[a]	0.9683	0.8882	0.7864	0.6950	0.6096
	H[b]	0.9668	0.8860	0.7818	0.6868	0.6008
	M[c]	0.9662	0.8837	0.7799	0.6854	0.6033
0.3	F[a]	0.9680	0.8851	0.7818	0.6876	0.6026
	H[b]	0.9662	0.8786	0.7812	0.6693	0.5811
	M[c]	0.9670	0.8789	0.7818	0.6825	0.5938
0.4	F[a]	0.9652	0.8783	0.7825	0.6773	0.5909
	H[b]	0.9603	0.8601	0.7449	0.6421	0.5492
	M[c]	0.9704	0.8872	0.7785	0.6731	0.5872

[a] First order theory results.
[b] Eleven-variable higher order theory results.
[c] Present seven-variable higher order theory results.

between the present theory results and the previous higher order theory results increase as Poisson's ratio increases. The FORTRAN programs with IBM PS-55 4MB RAM computer was used to simulate the nonlinear vibration problems. The ratio of the time required to analyse the typical example with 11 variables and seven-variable plate theory is nearly 9:7. From the data presented in Tables 1–4, it may be concluded that the present seven-variable higher order plate theory is simple and acceptable.

The effect of initial stresses on the frequency ratios of an initially stressed isotropic plate is clarified in Figs. 2 and 3 and Table 5. The plots of frequency ratio versus nondimensional vibration amplitude for different initial stress conditions are shown in Fig. 2. It is evident that the compressive edge load produces a softening effect on the frequency ratio and the tensile load has reverse effects. Under the compressive load, the results from the previous higher order theory show more softening effect than that of the present higher order theory as the amplitude is enlarged. This makes the frequency ratio predicted by these two theories show apparent difference at large amplitude vibration except when tensile load is applied. In Fig. 3, one can see that the linear frequencies of plates will reduce to zero when the compressive stress increases. The buckling load is obtained when the linear frequency is zero. In the whole range of initial stress calculated, the value of the frequency ratio predicted by the previous higher order theory is always smaller than the value from present theory especially under compressive load. Table 4 depicts how the value of the frequency

Table 5
ω_l/ω_{nl} for various values of the bending stress to normal stress ($a/b=1$, $a/h=10$, $W_{max}=1$, $\nu=0.3$)

		$\beta=0$	$\beta=5$	$\beta=10$	$\beta=15$	$\beta=20$	$\beta=25$	$\beta=30$
$k=2$	F[a]	0.681	0.681	0.681	0.681	0.681	0.681	0.680
	H[b]	0.672	0.672	0.672	0.673	0.674	0.673	0.669
	M[c]	0.680	0.679	0.680	0.681	0.678	0.679	0.678
$k=-2$	F[a]	0.488	0.462	0.458	0.457	0.457	0.456	0.451
	H[b]	0.306	0.305	0.303	0.296	0.294	0.291	0.285
	M[c]	0.455	0.441	0.420	0.414	0.416	0.410	0.408

[a] First order theory results.
[b] Eleven-variable higher order theory results.
[c] Present seven-variable higher order theory results.

ratio is changed as the value of the bending stress ratio β (bending stress to normal stress) varies. For the tensile normal stress condition, the increase in the bending stress ratio has little effect on the frequency ratios predicted by both theories. But, the larger bending stress ratio has the lower frequency ratio under compressive stress.

6. Summary

The preliminary results presented here indicate the following:

1. The displacement field of present seven-variable higher order plate theory satisfies the stress-free boundary conditions and the theory gives nearly similar results as the previous 11-variable higher order plate theory results.
2. For small amplitude and low Poisson's ratio, the frequency ratios have similar values for two higher order theories while at large amplitude and high Poisson's ratio the difference between them become more significant.
3. The frequency ratio decreases with the increasing vibration amplitude and Poisson's ratio, but it decreases with the decreasing thickness ratio and aspect ratio.
4. The initial compressive stresses significantly reduce the frequency ratio of the larger amplitude vibrations, and the tensile stresses have reverse effects.
5. At tensile condition, the frequency ratio slightly decreases with the increasing bending stress ratio. But the compressive condition has reverse effect.

Appendix A

$$\begin{bmatrix} [X_1] \\ [X_2] \\ Q_{11} \\ [X_3] \\ Q_{17} \\ [X_4] \end{bmatrix} = \begin{bmatrix} [A_1] & [B_1] & [A_2] & [D_1] & 2[B_2] & [E_1] \\ [B_1] & [D_1] & [B_2] & [E_1] & 2[D_2] & [F_1] \\ [A_3] & [B_3] & A_{33} & [D_3] & 2B_{33} & [E_3] \\ [D_1] & [E_1] & [D_2] & [F_1] & 2[E_2] & [G_1] \\ [B_3] & [D_3] & B_{33} & [E_3] & 2D_{33} & [F_3] \\ [E_1] & [F_1] & [E_2] & [G_1] & 2[F_2] & [H_1] \end{bmatrix} \begin{bmatrix} [T_1] \\ [T_2] \\ [\varphi_z] \\ [T_3] \\ [\xi_z] \\ [T_4] \end{bmatrix},$$

$$\begin{bmatrix} [X_5] \\ [X_6] \\ [X_7] \end{bmatrix} = \begin{bmatrix} [A_4] & [A_4] & [B_4] & 2[B_4] & [D_4] & 3[D_4] \\ [B_4] & [B_4] & [D_4] & 2[D_4] & [E_4] & 3[E_4] \\ [D_4] & [D_4] & [E_4] & 2[E_4] & [F_4] & 3[F_4] \end{bmatrix} [\![X_8]\!],$$

$$[\![X_8]\!] = [\![T_5] \; [T_6] \; [T_7] \; [T_8] \; [T_9] \; [T_{10}]\!]^{\mathrm{T}},$$

$$[X_1] = [\,Q_1 \quad Q_2 \quad Q_3\,]^{\mathrm{T}}, \qquad [X_5] = [\,Q_4 \quad Q_5\,]^{\mathrm{T}},$$

$$[X_2] = [\,Q_6 \quad Q_7 \quad Q_8\,]^{\mathrm{T}}, \qquad [X_6] = [\,Q_9 \quad Q_{10}\,]^{\mathrm{T}},$$

$$[X_3] = [\,Q_{12} \quad Q_{13} \quad Q_{14}\,]^{\mathrm{T}}, \qquad [X_7] = [\,Q_{15} \quad Q_{16}\,]^{\mathrm{T}},$$

$$[X_4] = [\,Q_{18} \quad Q_{19} \quad Q_{20}\,]^{\mathrm{T}},$$

$$[\Gamma_1] = \begin{bmatrix} \Gamma_{11} & 0 & 0 & \Gamma_{12} \\ 0 & \Gamma_{66} & \Gamma_{66} & 0 \\ \Gamma_{12} & 0 & 0 & \Gamma_{22} \end{bmatrix}, \qquad \Gamma = A, D, F, H,$$

$$[\Gamma_2] = [\,\Gamma_{13} \quad 0 \quad \Gamma_{23}\,]^{\mathrm{T}}, \qquad \Gamma = A, D, F,$$

$$[\Gamma_3] = [\,\Gamma_{13} \quad 0 \quad 0 \quad \Gamma_{23}\,], \qquad \Gamma = A, D, F,$$

$$[\Gamma_4] = \begin{bmatrix} \Gamma_{55} & 0 \\ 0 & \Gamma_{55} \end{bmatrix}, \qquad \Gamma = A, D, F,$$

$$[T_1] = [\,u_{x,x} \quad u_{x,y} \quad u_{y,x} \quad u_{y,y}\,]^{\mathrm{T}},$$

$$[T_2] = [\,\varphi_{x,x} \quad \varphi_{x,y} \quad \varphi_{y,x} \quad \varphi_{y,y}\,]^{\mathrm{T}},$$

$$[T_3] = -\tfrac{1}{2}[\,\varphi_{z,xx} \quad \varphi_{z,xy} \quad \varphi_{z,yx} \quad \varphi_{z,yy}\,]^{\mathrm{T}},$$

$$[T_4] = -\frac{4}{3h^2}\begin{bmatrix} w_{,xx} + \varphi_{x,x} + h^2\xi_{z,xx}/4 \\ w_{,xy} + \varphi_{x,y} + h^2\xi_{z,xy}/4 \\ w_{,yx} + \varphi_{y,x} + h^2\xi_{z,yx}/4 \\ w_{,yy} + \varphi_{y,y} + h^2\xi_{z,yy}/4 \end{bmatrix},$$

$$[T_5] = [\,w_{,x} \quad w_{,y}\,]^{\mathrm{T}}, \qquad [T_6] = [\,\varphi_x \quad \varphi_y\,]^{\mathrm{T}}, \qquad [T_7] = [\,\varphi_{z,x} \quad \varphi_{z,y}\,]^{\mathrm{T}},$$

$$[T_8] = \tfrac{1}{2}[\,\varphi_{z,x} \quad \varphi_{z,y}\,]^{\mathrm{T}}, \qquad [T_9] = [\,\xi_{z,x} \quad \xi_{z,y}\,]^{\mathrm{T}},$$

$$[T_{10}] = -\frac{4}{3h^2}\begin{bmatrix} w_{,x} + \varphi_x + (h^2/4)\xi_{z,x} \\ w_{,y} + \varphi_y + (h^2/4)\xi_{z,y} \end{bmatrix},$$

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \\ L_{38} \end{bmatrix} = \begin{bmatrix} A_{11} & D_{11} & F_{11} \\ D_{11} & F_{11} & H_{11} \\ F_{11} & H_{11} & K_{11} \\ A_{12} & D_{12} & F_{12} \\ D_{12} & F_{12} & H_{12} \\ F_{12} & H_{12} & K_{12} \\ A_{13} & D_{13} & F_{13} \end{bmatrix} \begin{bmatrix} w_{,x}^2/2 \\ \varphi_{z,x}^2/2 + w_{,x}\xi_{z,x} \\ \xi_{z,x}^2/2 \end{bmatrix},$$

$$\begin{bmatrix} L_{11} \\ L_{12} \\ L_{13} \\ L_{14} \\ L_{15} \\ L_{16} \\ L_{39} \end{bmatrix} = \begin{bmatrix} A_{12} & D_{12} & F_{12} \\ D_{12} & F_{12} & H_{12} \\ F_{12} & H_{12} & K_{12} \\ A_{22} & D_{22} & F_{22} \\ D_{22} & F_{22} & H_{22} \\ F_{22} & H_{22} & K_{22} \\ A_{23} & D_{23} & F_{23} \end{bmatrix} \begin{bmatrix} w_{,y}^2/2 \\ \varphi_{z,y}^2/2 + w_{,y}\xi_{z,y} \\ \xi_{z,y}^2/2 \end{bmatrix},$$

$$\begin{bmatrix} L_7 \\ L_8 \\ L_9 \\ L_{10} \end{bmatrix} = \begin{bmatrix} D_{11} & F_{11} \\ F_{11} & H_{11} \\ D_{12} & F_{12} \\ F_{12} & H_{12} \end{bmatrix} \begin{bmatrix} w_{,x}\varphi_{z,x} \\ \varphi_{z,x}\xi_{z,x} \end{bmatrix}, \qquad \begin{bmatrix} L_{17} \\ L_{18} \\ L_{19} \\ L_{20} \end{bmatrix} = \begin{bmatrix} D_{12} & F_{12} \\ F_{12} & H_{12} \\ D_{22} & F_{22} \\ F_{22} & H_{22} \end{bmatrix} \begin{bmatrix} w_{,y}\varphi_{z,y} \\ \varphi_{z,y}\xi_{z,y} \end{bmatrix},$$

$$\begin{bmatrix} L_{21} \\ L_{22} \\ L_{23} \\ L_{24} \end{bmatrix} = \begin{bmatrix} A_{66} & 0 & D_{66} & 0 & F_{66} \\ 0 & D_{66} & 0 & F_{66} & 0 \\ D_{66} & 0 & F_{66} & 0 & H_{66} \\ 0 & F_{66} & 0 & H_{66} & 0 \end{bmatrix} \begin{bmatrix} w_{,x}w_{,y} \\ w_{,x}\varphi_{z,y} + w_{,y}\varphi_{z,x} \\ \varphi_{z,x}\varphi_{z,y} + w_{,x}\xi_{z,y} + w_{,y}\xi_{z,x} \\ \varphi_{z,x}\xi_{z,y} + \varphi_{z,x}\xi_{z,y} \\ \xi_{z,x}\xi_{z,y} \end{bmatrix},$$

$$\begin{bmatrix} L_{26} \\ L_{27} \\ L_{28} \\ L_{29} \\ L_{30} \\ L_{31} \end{bmatrix} = \begin{bmatrix} Q_1 + L_1 + L_{11} & Q_6 + L_7 + L_{17} & Q_{12} + L_2 + L_{12} \\ Q_6 + L_7 + L_{17} & Q_{12} + L_2 + L_{12} & Q_{18} + L_8 + L_{18} \\ Q_{12} + L_2 + L_{12} & Q_{18} + L_8 + L_{18} & Q_{21} + L_3 + L_{13} \\ Q_2 + L_{21} & Q_7 + L_{24} & Q_{13} + L_{22} \\ Q_7 + L_{24} & Q_{13} + L_{22} & Q_{19} + L_{25} \\ Q_{13} + L_{22} & Q_{19} + L_{25} & Q_{22} + L_{23} \end{bmatrix} \begin{bmatrix} w_{,x} \\ \varphi_{z,x} \\ \xi_{z,x} \end{bmatrix},$$

$$\begin{bmatrix} L_{35} \\ L_{36} \\ L_{37} \\ L_{32} \\ L_{33} \\ L_{34} \end{bmatrix} = \begin{bmatrix} Q_3 + L_4 + L_{14} & Q_8 + L_9 + L_{19} & Q_{14} + L_5 + L_{15} \\ Q_8 + L_9 + L_{19} & Q_{14} + L_5 + L_{15} & Q_{20} + L_{10} + L_{20} \\ Q_{14} + L_5 + L_{15} & Q_{20} + L_{10} + L_{20} & Q_{23} + L_6 + L_{16} \\ Q_2 + L_{21} & Q_7 + L_{24} & Q_{13} + L_{22} \\ Q_7 + L_{24} & Q_{13} + L_{22} & Q_{19} + L_{25} \\ Q_{13} + L_{22} & Q_{19} + L_{25} & Q_{22} + L_{23} \end{bmatrix} \begin{bmatrix} w_{,y} \\ \varphi_{z,y} \\ \xi_{z,y} \end{bmatrix},$$

$$\begin{aligned}
N_a &= \tfrac{1}{2}(A_{13}w_{,x}^2 + D_{13}\varphi_{z,x}^2 + F_{13}\xi_{z,x}^2) + D_{13}w_{,x}\xi_{z,x} + \tfrac{1}{2}(A_{23}w_{,y}^2 + D_{23}\varphi_{z,y}^2 + F_{23}\xi_{z,y}^2 + D_{23}w_{,y}),\\
N_b &= 2(D_{13}w_{,x}\varphi_{z,x} + F_{13}\varphi_{z,x}\xi_{z,x} + D_{23}w_{,y}\varphi_{z,y} + F_{23}\varphi_{z,y}\xi_{z,y}),\\
N_c &= [A_{55}(\varphi_x + w_{,x}) + D_{55}(3\phi_x + \xi_{z,x})]w_{,x} + D_{55}(2\xi_x + \varphi_{z,x})\varphi_{z,x}\\
&\quad + [D_{55}(\varphi_x + w_{,x}) + F_{55}(3\phi_x + \xi_{z,x})]\xi_{z,x},\\
N_d &= D_{55}(2\xi_x + \varphi_{z,x})w_{,x} + [D_{55}(\varphi_x + w_{,x}) + F_{55}(3\phi_x + \xi_{z,x})]\varphi_{z,x} + F_{55}(2\xi_x + \varphi_{z,x})\xi_{z,x},\\
N_e &= [A_{44}(\varphi_y + w_{,y}) + D_{44}(3\phi_y + \xi_{x,y})]w_{,y} + D_{44}(2\xi_y + \varphi_{z,y})\varphi_{z,y}\\
&\quad + [D_{44}(\varphi_y + w_{,y}) + F_{44}(3\phi_y + \xi_{z,y})]\xi_{z,y},\\
N_f &= D_{44}(2\xi_y + \varphi_{z,y})w_{,y} + [D_{44}(\varphi_y + w_{,y}) + F_{44}(3\phi_y + \xi_{z,y})]\varphi_{z,y} + F_{44}(2\xi_y + \varphi_{z,y})\xi_{z,y},\\
L_{40} &= -(N_a + N_c + N_e), \qquad L_{41} = -N_b - 2(N_d + N_f),
\end{aligned}$$

$$\{O_1\} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix}, \quad \{O_2\} = \begin{Bmatrix} R_5 \\ R_6 \\ R_7 \\ R_8 \end{Bmatrix}, \quad \{O_3\} = \begin{Bmatrix} R_9 \\ R_{10} \\ R_{11} \\ R_{12} \end{Bmatrix}, \quad \{O_4\} = \begin{Bmatrix} R_{13} \\ R_{14} \\ R_{15} \\ R_{16} \end{Bmatrix},$$

$$\{O_5\} = \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{Bmatrix}, \quad \{O_6\} = \begin{Bmatrix} S_5 \\ S_6 \\ S_7 \\ S_8 \end{Bmatrix}, \quad \{O_7\} = \begin{Bmatrix} S_9 \\ S_{10} \\ S_{11} \\ S_{12} \end{Bmatrix}, \quad \{O_8\} = \begin{Bmatrix} S_{13} \\ S_{14} \\ S_{15} \\ S_{16} \end{Bmatrix},$$

$$\{O_9\} = \begin{Bmatrix} R_{17} \\ R_{18} \\ R_{19} \\ R_{20} \end{Bmatrix}, \qquad \{O_{10}\} = \begin{Bmatrix} S_{17} \\ S_{18} \\ S_{19} \\ S_{20} \end{Bmatrix}, \qquad \{O_{11}\} = \begin{Bmatrix} R_{21} \\ R_{22} \\ R_{23} \\ R_{24} \end{Bmatrix}, \qquad \{O_{12}\} = \begin{Bmatrix} S_{21} \\ S_{22} \\ S_{23} \\ S_{24} \end{Bmatrix},$$

$$\{O_{13}\} = \begin{Bmatrix} R_{30} \\ R_{31} \\ R_{32} \end{Bmatrix}, \qquad \{O_{14}\} = \begin{Bmatrix} R_{33} \\ R_{34} \\ R_{35} \end{Bmatrix}, \qquad \{O_{15}\} = \begin{Bmatrix} S_{30} \\ S_{31} \\ S_{32} \end{Bmatrix}, \qquad \{O_{16}\} = \begin{Bmatrix} S_{33} \\ S_{34} \\ S_{35} \end{Bmatrix},$$

$$\{O_{17}\} = \begin{Bmatrix} R_{25} \\ R_{26} \\ R_{27} \end{Bmatrix}, \qquad \{O_{18}\} = \begin{Bmatrix} S_{25} \\ S_{26} \\ S_{27} \end{Bmatrix}, \qquad \{O_{19}\} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}, \qquad \{O_{20}\} = \begin{Bmatrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix},$$

$$\{O_{21}\} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix}, \qquad \{O_{22}\} = \begin{Bmatrix} U_9 \\ U_{10} \\ U_{11} \\ U_{12} \end{Bmatrix}, \qquad \{O_{23}\} = \begin{Bmatrix} U_{13} \\ U_{14} \\ U_{15} \\ U_{16} \end{Bmatrix}, \qquad \{O_{24}\} = \begin{Bmatrix} W_5 \\ W_6 \\ W_7 \\ W_8 \end{Bmatrix},$$

$$\{O_{25}\} = \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix}, \qquad \{O_{26}\} = \begin{Bmatrix} V_4 \\ V_5 \\ V_6 \end{Bmatrix}, \qquad \{O_{27}\} = \begin{Bmatrix} W_9 \\ W_{10} \\ W_{11} \end{Bmatrix},$$

$$\{O_1\} = \left[\sum_{xx}\right]\{\Delta_{xx}\}, \qquad \{O_2\} = \left[\sum_{xy}\right]\{\Delta_{xy}\}, \qquad \{O_3\} = \left[\sum_{xx}\right]\{\Delta_{yx}\},$$

$$\{O_4\} = \left[\sum_{xy}\right]\{\Delta_{yy}\}, \qquad \{O_5\} = \left[\sum_{yy}\right]\{\Delta_{xy}\}, \qquad \{O_6\} = \left[\sum_{xy}\right]\{\Delta_{xx}\},$$

$$\{O_7\} = \left[\sum_{yy}\right]\{\Delta_{yy}\}, \qquad \{O_8\} = \left[\sum_{xy}\right]\{\Delta_{yx}\}, \qquad \{O_9\} = \left[\sum_{xz}\right]\{\Delta_{zx}\},$$

$$\{O_{10}\} = \left[\sum_{yz}\right]\{\Delta_{zx}\}, \qquad \{O_{11}\} = \left[\sum_{xz}\right]\{\Delta_{zy}\}, \qquad \{O_{12}\} = \left[\sum_{yz}\right]\{\Delta_{zy}\},$$

$$\{O_{13}\} = [\gamma_{xx}]\{\Delta_x\}, \qquad \{O_{14}\} = [\gamma_{xy}]\{\Delta_y\}, \qquad \{O_{15}\} = [\gamma_{xy}]\{\Delta_x\},$$

$$\{O_{16}\} = [\gamma_{yy}]\{\Delta_y\}, \qquad \{O_{17}\} = [\Omega_{xz}]\{\pi_z\}, \qquad \{O_{18}\} = [\Omega_{yz}]\{\pi_z\},$$

$$\{O_{19}\} = \left[\sum_{zx}\right]\{\Delta_{xx}\}, \qquad \{O_{20}\} = \left[\sum_{zy}\right]\{\Delta_{xy}\}, \qquad \{O_{21}\} = \left[\sum_{zz}\right]\{\Delta_{zx}\},$$

$$\{O_{22}\} = \left[\sum_{zx}\right]\{\Delta_{yx}\}, \qquad \{O_{23}\} = \left[\sum_{zy}\right]\{\Delta_{yy}\}, \qquad \{O_{24}\} = \left[\sum_{zz}\right]\{\Delta_{zy}\},$$

$$\{O_{25}\} = [\gamma_{zx}]\{\Delta_x\}, \qquad \{O_{26}\} = [\gamma_{zy}]\{\Delta_y\}, \qquad \{O_{27}\} = [\Omega_{zz}]\{\pi_z\},$$

$$\left[\sum_{ij}\right] = \begin{bmatrix} N_{ij} & M_{ij} & M^*_{ij} & P_{ij} \\ M_{ij} & M^*_{ij} & P_{ij} & P^*_{ij} \\ M^*_{ij} & P_{ij} & P^*_{ij} & R_{ij} \\ P_{ij} & P^*_{ij} & R_{ij} & R^*_{ij} \end{bmatrix}, \quad (i = x, y, z;\ j = x, y),$$

$$\left[\sum_{iz}\right] = \begin{bmatrix} N_{iz} & M_{iz} & M^*_{iz} \\ M_{iz} & M^*_{iz} & P_{iz} \\ M^*_{iz} & P_{iz} & P^*_{iz} \\ P_{iz} & P^*_{iz} & R_{iz} \end{bmatrix}, \quad (i = x, y, z),$$

$$[\gamma_{ij}] = \begin{bmatrix} N_{ij} & M_{ij} & M^*_{ij} \\ M_{ij} & M^*_{ij} & P_{ij} \\ M^*_{ij} & P_{ij} & P^*_{ij} \end{bmatrix}, \quad (i = x, y, z;\ j = x, y), \qquad [\Omega_{iz}] = \begin{bmatrix} N_{iz} & M_{iz} \\ M_{iz} & M^*_{iz} \\ M^*_{iz} & P_{iz} \end{bmatrix}, \quad (i = x, y),$$

$$\{\Delta_{ij}\} = \left\{ u_{i,j} \varphi_{i,j} - \frac{1}{2}\varphi_{z,ij} - \frac{4}{3h^2}(w_{,i} + \varphi_x + h^2\xi_{z,i}/4)_{,j} \right\}^{\mathrm{T}},$$

$$\{\Delta_{ji}\} = \left\{ \varphi_i - \varphi_{z,i} - \frac{4}{h^2}(w_{,i} + \varphi_x + h^2\xi_{z,i}/4) \right\}^{\mathrm{T}}, \quad (i = x, y;\ j = x, y, z),$$

$$\{\pi_i\} = \{\varphi_i - \varphi_{z,i}\}^{\mathrm{T}}, \quad (i = x, y),$$

$$f_i = \int (\overline{F}_i + \Delta F_i)\,\mathrm{d}z + g_i, \quad (i = x, y, z),$$

$$h_i = \int (\overline{F}_i + \Delta F_i)\,\mathrm{d}z + \frac{h^2}{4} g_i, \quad (i = x, y, z),$$

$$m_i = \int (\overline{F}_i + \Delta F_i)\,\mathrm{d}z + \frac{h^2}{4} g_{ii}, \quad (i = x, y, z),$$

$$q_i = \int (\overline{F}_i + \Delta F_i)\,\mathrm{d}z + \frac{h^3}{8} g_{ii}, \quad (i = x, y),$$

$$\begin{aligned} g_i = {} & u_{i,x}(\sigma^+_{31} - \sigma^-_{31}) + \frac{h}{2}\varphi_{i,x}(\sigma^+_{31} + \sigma^-_{31}) - \frac{h^2}{8}\varphi_{z,ix}(\sigma^+_{31} + \sigma^-_{31}) - \frac{h}{6}\left(w_{,ix} + \varphi_{i,x} + \frac{h^2}{4}\xi_{z,ix}\right)(\sigma^+_{31} + \sigma^-_{31}) \\ & + u_{i,y}(\sigma^+_{32} - \sigma^-_{32}) + \frac{h}{2}\varphi_{i,y}(\sigma^+_{32} + \sigma^-_{32}) - \frac{h^2}{8}\varphi_{z,iy}(\sigma^+_{32} + \sigma^-_{32}) - \frac{h}{6}\left(w_{,iy} + \varphi_{i,y} + \frac{h^2}{4}\xi_{z,iy}\right)(\sigma^+_{32} + \sigma^-_{32}) \\ & - \frac{h}{2}\varphi_{z,i}(\sigma^+_{33} + \sigma^-_{33}) - \left(w_{,i} + \frac{h^2}{4}\xi_{z,i}\right)(\sigma^+_{33} - \sigma^-_{33}) + (\sigma^+_{33} - \sigma^-_{33}), \end{aligned}$$

$$g_{ii} = u_{i,x}(\sigma_{31}^{+} + \sigma_{31}^{-}) + \frac{h}{2}\varphi_{i,x}(\sigma_{31}^{+} - \sigma_{31}^{-}) - \frac{h^2}{8}\varphi_{z,ix}(\sigma_{31}^{+} - \sigma_{31}^{-}) - \frac{h}{6}\left(w_{,ix} + \varphi_{i,x} + \frac{h^2}{4}\xi_{z,ix}\right)(\sigma_{31}^{+} - \sigma_{31}^{-})$$
$$+ u_{i,y}(\sigma_{32}^{+} + \sigma_{32}^{-}) + \frac{h}{2}\varphi_{i,y}(\sigma_{32}^{+} - \sigma_{32}^{-}) - \frac{h^2}{8}\varphi_{z,iy}(\sigma_{32}^{+} - \sigma_{32}^{-}) - \frac{h}{6}\left(w_{,iy} + \varphi_{i,y} + \frac{h^2}{4}\xi_{z,iy}\right)(\sigma_{32}^{+} - \sigma_{32}^{-})$$
$$- \frac{h}{2}\varphi_{z,i}(\sigma_{33}^{+} - \sigma_{33}^{-}) - \left(w_{,i} + \frac{h^2}{4}\xi_{z,i}\right)(\sigma_{33}^{+} + \sigma_{33}^{-}) + (\sigma_{33}^{+} + \sigma_{33}^{-}).$$

Appendix B

$$C_{1,1} = -(12 + A_{66}h^2R^2/D_{11} + K),$$

$$C_{1,2} = -(A_{12}h^2 + A_{66}h^2)R/D_{11}, \qquad C_{1,3} = -\beta K\pi/30r, \qquad C_{14} = -\beta K/6,$$

$$C_{1,6} = A_{13}h^2r/(\pi D_{11}) - \frac{\pi}{2r}(1 + D_{66}(1+R)R/D_{11} + D_{12}R/D_{11} + K/12),$$

$$C_{1,7} = -\beta K\pi/120r, \qquad C_{1,1} = -(12 + A_{66}h^2R^2/D_{11} + K),$$

$$C_{2,2} = -(A_{66} + A_{22}R^2)h^2/D_{11}, \qquad C_{2,3} = -\beta K\pi/30r, \qquad C_{2,5} = -\beta K/6,$$

$$C_{2,6} = A_{23}h^2Rr/(\pi D_{11}) - \frac{\pi}{2r}(D_{22}R^2 + D_{66}(1+R)/D_{11} + D_{12}R/D_{11} + K/12),$$

$$C_{2,7} = -\beta K\pi R/120r,$$

$$C_{3,3} = -(A_{55}h^2/D_{11} + A_{44}h^2R^2/D_{11} + K) + 4(D_{55} + D_{44}R)/D_{11},$$

$$C_{3,4} = -(A_{55}h^2r - 4D_{55}r)/(D_{11}\pi), \qquad C_{3,5} = -(A_{44}h^2R^2r - 4D_{44}Rr)/(D_{11}\pi),$$

$$C_{3,6} = -\beta K/6, \qquad C_{3,7} = -(D_{55}/D_{11} + D_{44}R^2/D_{11} + K/12) + (D_{55} + D_{44}R)/D_{11},$$

$$C_{4,4} = -(\tfrac{6}{5} + (D_{66}h^2 + F_{66})R^2/(h^2D_{11}) + (A_{55}h^2 + 3D_{55})r^2/(\pi^2D_{11}) + 23K/240),$$

$$C_{4,5} = A_{13}h^2r/(\pi D_{11}) - \tfrac{4}{3}(F_{12} + F_{66})R/(h^2D_{11}), \qquad C_{4,6} = \pi\beta K/80r,$$

$$C_{4,7} = -\left(\frac{1}{20} - 2rD_{13}/\pi + D_{55}\left(1 + \frac{r}{\pi}\right)\frac{r}{\pi} + \frac{1}{3}(F_{12} + F_{66}(1+R))R/(h^2D_{11}) + K/80\right),$$

$$C_{5,5} = -(D_{66} + D_{22}R^2 + (A_{44}h^2 + 4D_{11})r^2/\pi^2 + 23K/240 + \tfrac{4}{3}(F_{22}R^2 + F_{66})/h^2)/D_{11},$$

$$C_{5,7} - (2D_{13}R - D_{44}(1+R)Rr/\pi + \tfrac{1}{3}(F_{12} + F_{66}(1+R) + F_{22}R^2)\pi/rh^2)/D_{11} - K/240,$$

$$C_{6,6} = -D_{55}\left(1 + \frac{\pi}{2}\right) + D_{44}R\left(\frac{1}{2} + R\right) + A_{33}h^2r^2/\pi^2 + \pi(D_{13} + D_{23}R)/D_{11} + K/12,$$

$$C_{7,7} = -(2F_{55} + F_{44}(1+R)R - \tfrac{2}{3}(F_{13} + F_{23}R) + 4D_{33}r^2/\pi^2)/(h^2D_{11}) + Pk/80,$$

$$C_{6,7} = -\beta K/40, \qquad \tau = t(\pi^2 D_{11}/\rho h^3 a^2)^{1/2},$$

$$R = a/b, \quad r = a/h, \quad \beta = \sigma_\mathrm{m}/\sigma_\mathrm{n}, \quad K = N_{xx}h^2r^2/\pi^2R^2D_{11},$$

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \frac{16}{9\pi^2 r D_{11}} \begin{bmatrix} A_{11}h^2 & A_{12}h^2 & A_{66}h^2 \\ D_{11} & D_{12} & D_{66} \\ F_{11}/h^2 & F_{12}/h^2 & F_{66}/h^2 \\ H_{11}/h^4 & H_{12}/h^4 & H_{66}/h^4 \end{bmatrix} \begin{bmatrix} -2 \\ R^2 \\ -R^2 \end{bmatrix},$$

$$\begin{bmatrix} N_5 \\ N_6 \\ N_7 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{66} \\ F_{11}/h^2 & F_{12}/h^2 & F_{66}/h^2 \\ H_{11}/h^4 & H_{12}/h^4 & H_{66}/h^4 \end{bmatrix} \begin{bmatrix} -4 \\ 2R^2 \\ -R^2 \end{bmatrix},$$

$$\begin{bmatrix} N_8 \\ N_9 \\ N_{10} \\ N_{11} \end{bmatrix} = \frac{16}{9\pi^2 r D_{11}} \begin{bmatrix} A_{66}h^2 & A_{12}h^2 & A_{22}h^2 \\ D_{66} & D_{12} & D_{22} \\ F_{66}/h^2 & F_{12}/h^2 & F_{22}/h^2 \\ H_{66}/h^4 & H_{12}/h^4 & H_{22}/h^4 \end{bmatrix} \begin{bmatrix} -R \\ R \\ -2R^3 \end{bmatrix},$$

$$\begin{bmatrix} N_{12} \\ N_{13} \\ N_{14} \end{bmatrix} = \begin{bmatrix} D_{22} & D_{12} & D_{66} \\ F_{22}/h^2 & F_{12}/h^2 & F_{66}/h^2 \\ H_{22}/h^4 & H_{12}/h^4 & H_{66}/h^4 \end{bmatrix} \begin{bmatrix} -4R^3 \\ 2R \\ -R \end{bmatrix},$$

$$[N_{15} \quad N_{16} \quad N_{17} \quad N_{18}] = \tfrac{1}{\pi}[N_1 \quad N_2 \quad N_3 \quad N_4][-1 \quad 2 \quad 2]^\mathrm{T},$$

$$[N_{19} \quad N_{20} \quad N_{21} \quad N_{22}] = [N_8 \quad N_9 \quad N_{10} \quad N_{11}][2 \quad 2 \quad -1]^\mathrm{T},$$

$$\begin{bmatrix} N_{23} \\ N_{24} \\ N_{25} \\ N_{26} \\ N_{27} \end{bmatrix} = -\frac{\pi^2}{32 r^2 D_{11}} \begin{bmatrix} A_{11}h^2 & A_{12}h^2 & A_{66}h^2 & A_{22}h^2 \\ D_{11} & D_{12} & D_{66} & D_{22} \\ F_{11}/h^2 & F_{12}/h^2 & F_{66}/h^2 & F_{22}/h^2 \\ H_{11}/h^4 & H_{12}/h^4 & H_{66}/h^4 & H_{22}/h^4 \\ K_{11}/h^8 & K_{12}/h^8 & K_{66}/h^8 & K_{22}/h^8 \end{bmatrix} \begin{bmatrix} 9 \\ 2R^2 \\ 4R^4 \\ 9R^4 \end{bmatrix},$$

$$\begin{bmatrix} M_{28} \\ N_{29} \\ N_{30} \\ N_{31} \\ N_{32} \\ N_{33} \end{bmatrix} = -\frac{16}{9\pi^2 D_{11}} \begin{bmatrix} A_{13}h^2 & A_{23}h^2 \\ D_{13} & D_{23} \\ F_{13}/h^2 & F_{23}/h^2 \\ A_{55}h^2 & A_{44}h^2 \\ D_{55} & D_{44} \\ F_{55}/h^2 & F_{44}/h^2 \end{bmatrix} \begin{bmatrix} 1 \\ R^2 \end{bmatrix}, \qquad \begin{bmatrix} N_{34} \\ N_{35} \end{bmatrix} = -\frac{32r}{9\pi^3 D_{11}} \begin{bmatrix} D_{44} \\ D_{44}R \end{bmatrix},$$

$$N_{28} = M_{28} + 4\pi(N_6 + N_{13}R)/r, \qquad N_{36} = 4\pi(N_7 + N_{14}R)/r,$$

$$N_{37} = \pi(N_2 + N_9 R)/2r,$$

$$N_{38} = \pi(N_3 + N_{10}R)/2r, \qquad N_{39} = \pi(N_4 + N_{11}R)/2r, \qquad N_{40} = N_5 + 4N_6/3,$$

$$N_{41} = N_{12} + 4N_{13}/3, \qquad N_{39} = N_6 + 4N_7/3, \qquad N_{40} = N_{13} + 4N_4/3,$$

$$\bar{I}_i = I_i - \frac{4}{3h^2}I_{i+2}, \qquad R_{1,1} = R_{2,2} = I_1, \qquad R_{1,6} = \frac{\pi}{2r}I_3, \qquad R_{2,6} = \frac{\pi R}{2r}I_3,$$

$$R_{3,3} = I_1 + \frac{16\pi^2}{9r^2}(1+R)I_7, \qquad R_{3,4} = -\frac{4\pi}{3r}\bar{I}_5, \qquad R_{3,5} = -\frac{4\pi R}{3r}\bar{I}_5,$$

$$R_{3,7} = I_3 + \tfrac{4}{9}(1+R^2)I_7,$$

$$R_{4,4} = R_{5,5} = \bar{I}_3 - \tfrac{4}{3}\bar{I}_5, \qquad R_{4,7} = -\frac{\pi}{3r}\bar{I}_5, \qquad R_{5,7} = -\frac{\pi R}{3r}\bar{I}_5, \qquad R_{6,6} = \frac{\pi^2}{4r^2}(1+R^2)I_5,$$

$$R_{7,7} = I_5 + \frac{\pi^2}{9r^2}(1+R^2)I_7.$$

References

Ahmed, N.U., 1994. Higher-order finite element modelling of laminated composite plates. Int. J. Num. Meth. Engng. 37, 123–139.

Berger, H.M., 1955. New approach to the analysis of large deflections of plates. J. Appl. Mech. 22, 465–472.

Bert, C.W., 1984. A critical evaluation of new plate theories applied to laminated composites. Comp. Struct. 2, 329–347.

Bhimaraddi, A., Stevens, L.K., 1984. A higher order theory for free vibration of orthotropic, homogeneous and laminated rectangular plates. J. Appl. Mech. 51, 195–198.

Bolotin, V.V., 1963. Nonconservative problems of the theory of elastic stability. Macmillan, New York.

Brunelle, E.J., Robertson, S.R., 1974. Initially stressed Mindlin plate. AIAA J. 12, 1036–1045.

Chauhan, A.P., Ashwell, D.G., 1969. Small and large amplitude free vibrations square shallow shells. Int. J. Mech. Sci. 11, 337–350.

Chen, C.S., Lai, J.Y., Doong, J.L., 1994. Large amplitude vibration of plates according to a higher order deformation. J. Sound Vib. submitted for publication.

Chen, L.W., Doong, J.L., 1983a. Large amplitude vibration of an initially stressed moderately thick plate. J. Sound Vib. 89, 499–508.

Chen, L.W., Doong, J.L., 1983b. Large amplitude vibration of an initially stressed thick circular plate. AIAA. J. 21, 1317–1324.

Chia, C.Y., Parbhakrar, M.K., 1978. A general mode approach to nonlinear flexural vibrations of laminated rectangular plates. J. Appl. Mech. 45, 623–628.

Doong, J.L., 1987. Vibration and stability of an initially stressed thick plate according to a higher order deformation theory. J. Sound Vib. 113, 425–440.

Doong, J.L., Chen, T.J., Chen, L.W., 1987. Vibration and stability of an initially stressed laminated plate based on a higher order deformation theory. Comp. Struct. 7, 285–310.

Doong, J.L., Chen, C.S., 1988. Large amplitude vibration of a beam based on a higher-order deformation theory. Appl. Acous. 25, 281–293.

Doong, J.L., Lee, C., 1991. Vibration and stability of laminated plates based on a modified plate theory. J. Sound Vib. 151, 193–201.

Huang, C.L., 1972. Nonlinear axisymmetric flexural vibration equations of a cylindrical anisotropic circular plate. AIAA J. 10, 1378–1379.

Lo, K.H., Christensen, R.M., Wu, E.M., 1977a. A higher order theory of plate deformation, part I: homogeneous plates. J. Appl. Mech. 44, 663–668.

Lo, K.H., Christensen, R.M., Wu, E.M., 1977b. A higher order theory of plate deformation, part II: laminated plates. J. Appl. Mech. 44, 669–676.

Murth, A.V.K., Vellaichamy, S., 1987. On the higher order shear deformation theory of laminated composite panels. Comp. Struct. 8, 247–270.

Pillai, S.R.R., Rao, B.N., 1991a. Large amplitude vibration of thin plates. J. Sound Vib. 149, 509–512.

Pillai, S.R.R., Rao, B.N., 1991b. Improved solution for the nonlinear vibration of simply supported rectangular cross-ply plates. J. Sound Vib. 150, 517–519.

Raju, K.K., 1980. Nonlinear vibrations of thick plates using Mindlin plate elements. Int. J. Num. Meth. Engng. 15, 249–257.

Rao, B.N., 1992. Large amplitude free vibrations of simply supported uniform beams with immovable ends. J. Sound Vib. 155, 523–527.

Reddy, J.N., Huang, C.L., Singh, I.R., 1981. Large deflections and large amplitude vibrations of axisymmetric circular plates. Int. J. Num. Meth. Engng. 17, 527–541.

Reddy, J.N., Chao, W.C., 1982. Nonlinear oscillations of laminated, anisotropic, rectangular plates. Trans ASME 49, 396–402.

Reddy, J.N., 1984a. A refined nonlinear theory of plates with transverse shear deformation. Int. J. Solid Struct. 20, 881–896.

Reddy, J.N., 1984b. A simply higher order theory for laminated composite plates. J. Appl. Mech. 51, 745–752.

Reddy, J.N., Phan, N.D., 1985. Stability and vibration of isotropic orthotropic and laminated plates according to a higher-order shear deformation theory. J. Sound Vib. 98, 157–170.

Sathymoorthy, M., 1979. Effects of large amplitude shear and rotatory inertia on vibration of rectangular plates. J. Sound Vib. 63, 161–167.

Singh, G., Rao, G.V., Iyengar, N.G.R., 1991. Large amplitude free vibration of simply supported antisymmetric cross-ply plates. AIAA J. 29, 784–790.

Whitney, J.M., Sun, C.T., 1973. A higher order theory for extension motions of laminated plates. J. Sound Vib. 30, 85–97.

Wu, C.I., Vinson, J.R., 1969. Influences of large amplitude transverse shear deformation and rotatory inertia on lateral vibrations of transversely isotropic plates. J. Appl. Mech. 36, 254–260.